Appendix 2.3

1. **Inverse Function Theorem** It may look impossible to remember the choice of
$$\delta = \min\left(k - f(g(k) - \varepsilon)\,,\, f(g(k) + \varepsilon) - k\right) \tag{2}$$
in the proof of this Theorem. But a little Rough Work might help, starting by looking at *what we want*, namely $|g(y) - g(k)| < \varepsilon$ for y in an interval **centred** at k, i.e. $|y - k| < \delta$ for some $\delta > 0$. Open what we want out as
$$g(k) - \varepsilon < g(y) < g(k) + \varepsilon.$$
Apply f to all sides and, since f is strictly increasing, we get
$$f(g(k) - \varepsilon) < f(g(y)) < f(g(k) + \varepsilon)\,.$$
Yet f and g are inverses so
$$f(g(k) - \varepsilon) < y < f(g(k) + \varepsilon)\,.$$
Thus we have an interval of y which will satisfy $|g(y) - g(k)| < \varepsilon$. This interval contains k because $g(k) - \varepsilon < g(k) < g(k) + \varepsilon$ and thus, from applying f throughout,
$$f(g(k) - \varepsilon) < k < f(g(k) + \varepsilon)\,.$$
Consider now the general situation of a closed interval $[a, b]$ with a interior point $c \in (a, b) \subseteq [a, b]$. To get an interval centred on c inside $[a, b]$ we measure the distances of c from the end points and choose the smallest, i.e. $d = \min(c - a, b - c)$. Then
$$c \in [c - d, c + d] \subseteq [a, b]\,,$$
and we have an interval centred on c. In our case $c = k$, $a = f(g(k) - \varepsilon)$ and $b = f(g(k) + \varepsilon)$ when $d = \min(c - a, b - c)$ becomes (2).

2. **The inverse is continuous at the end points.** To show that the function g constructed in the Inverse Function Theorem is continuous at the *end points* of $[c, d]$ was left to the students. Here I supply the details for the left hand end point

   Proof To prove $\lim_{y \to c+} g(y) = g(c)$ let $\varepsilon > 0$ be given. Choose $\delta = f(g(c) + \varepsilon) - c > 0$.

Assume $c < y < c+\delta = f(g(c)+\varepsilon)$. Apply g throughout, an increasing function, so

$$\begin{aligned} g(c) < g(y) &< g(f(g(c)+\varepsilon)) \\ &= (g \circ f)(g(c)+\varepsilon) \\ &= g(c)+\varepsilon, \end{aligned}$$

since f and g are inverses of each other. Thus

$$c < y < c+\delta \implies |g(y)-g(c)| < \varepsilon,$$

that is, we have verified the definition of $\lim_{y\to c+} g(y) = g(c)$.

I leave it to the interested Student to show that $\lim_{y\to d-} g(y) = g(d)$. ■